# Local-Connectivity and Maximal Local-Connectivity on the class of Matching Composition Networks

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***Abstract*—The local connectivity of two vertices is defined as the maximum number of internally vertex-disjoint paths between them. In this paper, we define two vertices to be maximally local-connected, if the maximum number of internally vertex-disjoint paths between them equals the minimum degree of these two vertices. We prove that a (k + 1)-regular Matching Composition Network is maximally local-connected, even if there are at most (k − 1) faulty vertices in it.**

***Index Terms*—Interconnection networks, connectivity, local connectivity, matching composition network.**

## I. Introduction

An interconnection network is usually denoted as an undirected graph. The connectivity is a major parameter describing the connection status of a graph, which is defined as the minimum number of vertices whose removal results in a disconnected or trivial graph. A classical theorem of Menger [9] concerning the connectivity provides a local point of view, and the local concept was introduced: the local connectivity of two vertices in a graph is defined as the maximum number of internally vertex-disjoint paths between them. Following this concept, Volkman [17] discussed some issues on it, Oh et al. [10][11] and Shih et al. [14][15] investigated some related properties on the star graph and the class of hypercube-like networks, respectively. In recent days, such issue is discussed on many interconnection networks.

The Matching Composition Network (MCN) [3][8], a recursively constructed Topology is a family of interconnection networks. The construction of an MCN is to join two graphs $G_0$ and $G_1$ of the same number of vertices by adding a perfect matching between the vertices of $G_0$ and $G_1$. Many well-known interconnection networks are special cases of the MCN family, such as the Hypercube, [13] the Crossed cubes [6], the Twisted cubes [1][7], and the Möbius cubes [4].

In this paper, we investigate the property of local connectivity on the Matching Composition Network. Let $G$ be a graph, $x$ and $y$ be two distinct vertices in $G$ and $k= min\{deg(x), deg(y)\}$. We say that $x$ and $y$ are maximally local-connected, if there exist $k$ vertex-disjoint paths connecting $x$ and $y$. A graph $G$ is maximally local-connected if every pair of vertices in $G$ are maximally local-connected. A regular MCN is actually maximally local-connected. Moreover, even with a set of faulty vertices, we will propose a strong fault-tolerant version of maximal local-connectivity, and prove that a $(k + 1)$-regular MCN is $(k - 1)$-fault-tolerant maximally local-connected.

## II. Preliminary

The architecture of a multiprocessor system is usually modeled as an undirected graph. For the graph definitions and notations we follow [2]. Let $G = (V, E)$ be a graph, we use $V(G)$ and $E(G)$ to denote the vertex set $V$ and the edge set $E$, respectively. The *connectivity* of a graph $G$, written $\kappa(G)$, is the minimum size of a vertex set $S$ such that $G - S$ is disconnected or has only one vertex. A graph $G$ is *k-connected* if its connectivity is at least $k$. The degree of a vertex $x$ is the number of edges incident with it. We use $deg_G(x)$, or simply $deg(x)$ if there is no ambiguity, to denote the degree of vertex $x$ in $G$; and use $\delta(G)$ to denote the minimum degree of all the vertices in $G$. We say that $G$ is *maximally connected* if $\kappa(G) = \delta(G)$. Let $u$ and $v$ be two distinct vertices, a path $P$ between them is a sequence of adjacent vertices, $\langle u, w_1, w_2, \ldots, w_k, v\rangle$, where $w_1, w_2, \ldots, w_k$ are distinct ones. The *local connectivity* between two distinct vertices $u$ and $v$ is the maximum number of internally disjoint $u - v$ paths.

A pair of vertices $x$ and $y$ is *maximally local*-connected if the local connectivity of $x$ and $y$ equals $min\{deg(x), deg(y)\}$, and a graph $G$ is *maximally local-connected* if every pair of vertices in $G$ are maximally local-connected.

Let $G_0$ and $G_1$ be two graphs with the same number of vertices, and $M$ be an arbitrary perfect matching between $V(G_0)$ and $V(G_1)$. We use $G(G_0, G_1; M)$ to denote the Matching Composition Network composed of $G_0$ and $G_1$ by $M$, which has the vertex set $V(G) = V(G_0) \cup V(G_1)$ and the edge set $E(G) = E(G_0) \cup E(G_1) \cup M$.

Let $G$ be a graph, and $F$ be a subset of vertices, $F \subset V(G)$, the induced subgraph obtained by deleting the vertices of $F$ from $G$ is denoted by $G - F$. Let $u$ be a vertex, we use $N_G(u)$,

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or simply $N(G)$ if there is no ambiguity, to denote the set of vertices adjacent to $u$ in $G$. Let $V'$ be a set of vertices, the neighborhood of $V'$ is defined as the set $N_G(V') = \{\bigcup_{v\in V'} N_G(v)\} - V'$. A graph $G$ is *k-regular* if the degree of every vertex in $G$ is $k$, and graph $G$ is triangle-free if there is no cycle of length three.

In the remaining of this section, we introduce several lemmas which will be used to prove our main results in the following sections.

**Lemma 1.** *Let $G = (V, E)$ be a k-regular and triangle-free graph, and every two vertices in G have at most two common neighboring vertices. For every subset $V'$ of V with $|V'| = 2$, the number of neighbors* of $V'$ *is at least* $2k - 2$. *That is,* $|N_G(V')| \geq 2k - 2$.

**Lemma 2.** *Let G be a k-regular and triangle-free graph with n* vertices. *Then* $n \geq 2k$.

Below is a lemma stating the structural properties of a matching composition network. It shows that an MCN constructed by two $k$-regular subgraphs is quite fault resistant, that is, even with up to $2k - 1$ vertex faults present and the resulting graph disconnected, it will have a large connected component and exactly one small component, which is an isolated vertex.

**Lemma 3.** *Let $G_0$ and $G_1$ be two k-regular, maximally connected and triangle-free graphs with the same number of vertices, and let M be an arbitrary perfect matching between $G_0$ and $G_1$. Let $G = G(G_0, G_1; M)$ be a Matching Composition Network composed of $G_0$ and $G_1$, and let T be a set of vertices in G with* $|T| \leq 2k - 1$. *Assume that every two vertices in $G_i$, $i$ = 0, 1, have at most two common neighboring vertices, for all* $k \geq 1$. *Then $G - T$ satisfies that either (1) $G - T$ is connected or (2) $G - T$ has two connected components, one of which is a trivial component.*

In the above lemma, changing the condition of $T$ slightly by replacing a faulty vertex by a faulty edge, the connection status of an MCN remains the same.

**Lemma 4.** *Let $G_0$ and $G_1$ be two k-regular, maximally connected and triangle-free graphs with the same number of vertices, and let M be an arbitrary perfect matching between $G_0$ and $G_1$. Assume that every two vertices in $G_i$, i =* 0, 1, *have at most two common neighboring vertices, for all* $k \geq 1$. *Let $G = G(G_0, G_1; M)$ be a Matching Composition Network composed of $G_0$ and $G_1$, and $e_f$ be an edge in G and $T_v$ be a set of vertices in G with $|T_v| \leq 2k - 2$. Then $G - T_v - \{e_f\}$ satisfies either that (1) $G - T_v - \{e_f\}$ is connected or that (2) $G - T_v - \{e_f\}$ has two connected components, one of which is a trivial component.*

We make some remarks concerning the above lemmas. If both graphs $G_0$ and $G_1$ have the properties that (1) each one is triangle-free and (2) every pair of distinct vertices in each graph share at most two common neighbors, then the constructed MCN $G = (G_0, G_1; M)$ also has properties (1) and (2). Therefore the result can be applied recursively. We observe that many interconnection networks have these two properties. For example, the hypercube-like graphs and the star graphs do.

## III. Maximal Local-Connectivity

In this section, we discuss the maximal local-connectivity. A classical theorem about local-connectivity was provided by Menger as follows.

**Theorem 1.** *[9] Let x and y be two nonadjacent vertices of graph G. The minimum size of an x,y-cut equals the maximum number of pairwise internally disjoint x,y-paths.*

A graph $G$ is defined to be maximally local-connected if, for each pair of vertices $x$ and $y$, there are $min\{deg(x), deg(y)\}$ vertex-disjoint paths connecting them. We prove that for every two graphs $G_0$, $G_1$, not necessarily maximally local-connected, the generated MCN $G = G(G_0, G_1; M)$ is maximally local-connected. Moreover, even if the MCN contains a number of faulty vertices, the remaining graph is also maximally local-connected, provided that the number of faulty vertices is no greater than the minimum degree minus two.

Now we give the definition of a graph to be $f$-fault-tolerant maximally local-connected.

**Definition 1.** *A graph G is f-fault-tolerant maximally local-connected, abbreviated as f-maximally local-connected, if for a set of faulty vertices F, $|F| \leq f$, each pair of vertices x, y of $G - F$ are connected by $min\{deg_{G-F}(x), deg_{G-F}(y)\}$ vertex-disjoint fault-free paths, where $deg_{G-F}(x)$ and $deg_{G-F}(y)$ are the degrees of x and y in $G - F$, respectively.*

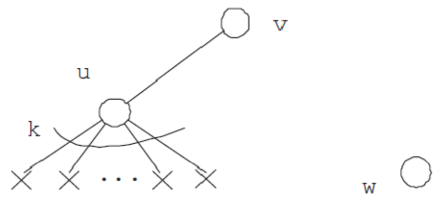


Fig. 1. An example showing that a ($k$+1)-regular MCN is not $k$-maximally local-connected.

In this section, we are going to prove that an MCN composed of two $k$-regular graphs with some additional properties is $(k - 1)$-maximally local-connected. Note that the MCN here is ($k$+1)-regular. This result is optimal in the sense that the result cannot be guaranteed if there are $k$ faulty vertices. We give an example to show it (as illustrated in Fig. 1), let $(u; v)$ be an edge in the MCN. Suppose that all the $k$ vertices adjacent to $u$ except $v$ are faulty. Choose a vertex $w$ different from $u$ and $v$, and $deg_{G-F}(v) = deg_{G-F}(w) = k + 1$. However, there are at most $k$ vertex-disjoint paths between $v$ and $w$. So the ($k$ +1)-regular MCN is not $k$-maximally local-connected. Before proving the main result, we make some simple observations.

If an MCN $G = G(G_0, G_1; M)$ is $(k - 1)$-maximally local-connected, the number of vertices in each component $G_i$, $i$ = 0, 1, has to be large enough. More precisely, each component $G_i$ has to contain at least $2k$ vertices.

Intuitively, if each component $G_i$ contains only $2k - 1$ or less vertices, there are at most $2k - 1$ "bridges" connecting $G_0$ and $G_1$ in the MCN $G = (G_0, G_1; M)$. If there are $k - 1$ faulty vertices to destroy $k - 1$ "bridges", there are only $k$ "bridges" left between $G_0$ and $G_1$. Pick a vertex $u$ in $G_0$ and another vertex $v$ in $G_1$, each with degree $k + 1$. Then it is intuitively clear that there are no $k + 1$ vertex-disjoint paths connecting $u$ and $v$. A formal proof is given below.

**Lemma 5.** *Let $G = G(G_0, G_1; M)$ be a Matching Composition Network composed of two k-regular graphs $G_0$ and $G_1$ both with the same number of vertices n, where $k \neq 2$. If G is $(k-1)$-maximally local-connected, then $n \geq 2k$.*

**Proof.** Before proving this Lemma, we explain why the lemma does not hold if $k = 2$.

When $k = 2$, graphs $G_0$ and $G_1$ are cycles of the same length. It is straightforward but tedious that the MCN generated here is 1-fault-tolerant maximally local-connected. However, it is not necessarily that $n \geq 2k$. For example $n = 3$, both $G_0$ and $G_1$ are triangles, and the number $n(= 3)$ is less than $2k(= 4)$.

When $k = 1$, graphs $G_0$ and $G_1$ are both one edge incident with two vertices. The MCN is indeed 0-fault-tolerant maximally local-connected and it also holds that $n \geq 2k$.

Now we consider the situation that $k \geq 3$. Suppose on the contrary that $n \leq 2k - 1$. If one of the two subgraphs $G_0$ and $G_1$ is a complete graph, since each subgraph is $k$-regular, the number of vertices in each subgraph is $k + 1$. Then both $G_0$ and $G_1$ are complete graph. Hence, the cardinality of the perfect matching $M$ between $V(G_0)$ and $V(G_1)$ is $k + 1$. Let $x$ be a vertex in $G_0$, and $y$ be the adjacent vertex of $x$ in $G1$. We choose a set of $k - 1$ vertices $V_f = \{f_1, f_2, \ldots, f_k - 1\}$ not containing $x$ and $y$, where $f_1$ is an arbitrary vertex in $G_0$, and $f_2, f_3, \ldots, f_k - 1$ are other vertices in $G_1$ not adjacent to $f_1$. In the induced subgraph of $V(G) - V_f$, the number of edges with one end in $G_0$ and the other end in $G_1$ is two, and the remaining degrees of $x$ and $y$ are $k$ and three, respectively. There are only two fault-free edges between $G_0$ and $G_1$. Therefore, it is easy to see that there does not exist three vertex-disjoint paths between $x$ and $y$, and the graph $G$ is not $(k-1)$-fault-tolerant maximally local-connected.

Now, suppose that neither $G_0$ nor $G_1$ is a complete graph. Let $x$ be a vertex in $G_0$, and $y$ be the adjacent vertex of $x$ in $G_1$. We index the adjacent vertices of $x$ in $G_0$ as $u_1, u_2, \ldots, u_k$. The other ones left in $G_0$ are indexed as $u_{k+1}, u_{k+2}, \ldots, u_{n-1}$, in any arbitrary order. For each $u_i$ in $G_0$, the corresponding (adjacent) vertex in $G_1$ is named as $v_i$, for $1 \leq i \leq k$. Since $G_1$ is not a complete graph and $G_1$ is $k$-regular, there exist two vertices $v_p, v_q \in \{v_1, v_2, \ldots, v_k\} \cup \{y\}$, such that $(v_p, v_q) \notin E(G_1)$. Without loss of generality, let $v_q \neq y$. Recall that $n \leq 2k - 1$. Now we pick vertices $\{u_i \mid k+1 \leq i \leq n-1\}$ and $v_q$ to form a vertex set $V_f$, which has cardinality at most $k-1$. The cardinality of $V_f$ can be verified as $[(n-1)-k]+1 \leq [((2k-1)-1)-k]+1 = k-1$. In the induced subgraph of $V(G)-V_f$, the remaining degrees of $x$ and $v_p$ are both $k + 1$. However, in this induced subgraph, there are only $k$ edges connecting the vertices in $G_0-V_f$ and $G_1-V_f$, which results in that there are no $k + 1$ vertex-disjoint paths between $x$ and $v_p$. So the graph $G$ is not $(k - 1)$-maximally local-connected.
□

Therefore, to study the $(k-1)$-fault-tolerant maximally local-connectivity of MCN, we need a $k$-regular graph containing at least $2k$ vertices. Recall that Lemma 2 states that every $k$-regular and triangle-free graph contains at least $2k$ vertices.

Now, we are ready to present our first main result.

**Theorem 2.** *Let $G_0$ and $G_1$ be two k-regular, maximally connected and triangle-free graphs with the same number of vertices, for $k \geq 1$, and let M be an* arbitrary *perfect matching between $G_0$ and $G_1$. Assume that any two vertices in $G_i$, $i = 0, 1$, have at most two common neighboring vertices. The Matching Composition Network $G = (G_0, G_1; M)$ is $(k-1)$-maximally local-connected.*

**Proof.** By Lemma 2, the number of vertices in $G_i$ is greater than $2k$, for $i = 0; 1$. Let $F$ be a set of faulty vertices with $|F| \leq k - 1$, and let $x$ and $y$ be two fault-free vertices in $G - F$. We assume without loss of generality that $deg_{G-F}(x) \leq deg_{G-F}(y)$, so $min\{deg_{G-F}(x), deg_{G-F}(y)\} = deg_{G-F}(x)$. We now show that after deleting $deg_{G-F}(x) - 1$ arbitrary vertices in $G - F$, vertex $x$ is still connected to $y$. By Theorem 1, this implies that each pair of vertices $x$ and $y$ are connected by $deg_{G-F}(x)$ vertex-disjoint fault-free paths, where $|F| \leq k - 1$. We now consider two cases:

**Case I:** *x and y are not adjacent in $G - F$.* We then show that $x$ is connected to $y$ if the number of vertices deleted is smaller than $deg_{G-F}(x)-1$. For the sake of contradiction, suppose that $x$ and $y$ are separated by deleting a set of vertices $V_f$, where $|V_f| \leq deg_{G-F}(x)-1$. As a consequence, $|V_f| \leq k$ because of $deg_{G-F}(x) \leq deg(x) \leq k + 1$. Then, the summation of the cardinality of these two sets $F$ and $V_f$ is $|F| + |V_f| \leq 2k - 1$. Let $T=F \cup V_f$. By Lemma 3, either $G-T$ is connected, or $G-T$ has two components, one of which contains only one vertex. If $G-T$ is connected, it contradicts to the assumption that $x$ and $y$ are disconnected. Otherwise, if $G-T$ has two components and one of which contains only one vertex $u$. Since we assume that $x$ and $y$ are separated, one of $x$ and $y$ is the vertex $u$, say $x = u$. Thus, the set $V_f$ must be the neighborhood of $x$ and $|V_f| = \mathrm{deg}_{G-F}(x)$, which is also a contradiction. Then, $x$ is connected to $y$ when the number of vertices deleted is smaller than $deg_{G-F}(x)-1$ in $G-F$.

**Case II:** *x and y are adjacent in $G - F$.* We need to show that $x$ is connected to $y$ if the number of vertices deleted is smaller than $deg_{G-F}(x)$ 2 in $G-F-\{(x, y)\}$. Suppose on the contrary that in $G-F-\{(x; y)\}$, $x$ and $y$ are separated by deleting a set of vertices $V_f$, where $|V_f| \leq deg_{G-F}(x) - 2$. Since $deg_{G-F}(x) \leq k + 1$, we get $|V_f| \leq k - 1$. Then the union set $T$ of $F$ and $V_f$ has cardinality $|T| = |F| + |V_f| \leq 2k - 2$. In this circumstance, Lemma 4 implies either that $G-T-\{(x, y)\}$ is connected or that $G-T-\{(x, y)\}$ has two components one of which is trivial. For the first situation, $G-T-\{(x, y)\}$ is connected, this contradicts to the assumption that $x$ and $y$ are separated in $G-F-\{(x; y)\}$. For the second situation, the trivial graph must be $x$ or $y$, and we without loss of generality let $x$ be such trivial graph. In $G-F-\{(x, y)\}$, in order to make $x$ a trivial graph, the number of vertices deleted must be greater or equal to $deg_{G-F}(x) -1 = k$. However, $|V_f| \leq k - 1$, which is a contradiction. So $x$ and $y$ are connected when the number of vertices deleted is smaller than $deg_{G-F}(x) - 2$ in $G-F-\{(x, y)\}$.

## IV. Conclusions

The fault tolerance is one of the important properties of network performance. In this paper, we prove that a $(k+1)$-regular Matching Composition Network is maximally local-connected, $(k-1)$-fault-tolerant maximally local-connected. Based on the generalized versions of connectivity proposed in this paper, the fault-tolerant capability may be increased if we add some restrictions on the Matching Composition Networks. In other words, if we add some conditions to the faulty vertices of the MCN, the upper bound of fault-tolerance may possibly be increased to exceed $k-1$. In addition to the graphs introduced in this paper, there are other interesting graphs.

They may have some similar strong connectivity properties as defined in this paper. These are issues worth studying.

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