A Survey of the Development of the Theory of Categories

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***Abstract*—The paper outlines a crisp and critical survey of the development of the theory of categories. We show that the theory originally arose in mathematics out of the need of formalism to describe the passage from one type of mathematical structure to another, and later became an autonomous field of research that has now occupied a central position in most of the branches of mathematics, some areas of theoretical computer science and mathematical physics.**

***Index Terms*—Axiomatic, category, computational.**

I. Introduction

In 1942, Samuel Eilenberge and Saunders Maclane presented a paper introducing *Specific Functors and Natural Transformation at work*, particularly confined to the study of groups. Later in 1945, they presented another paper titled *General Theory of Natural Equivalence* introducing Categories, Functors and Natural Transformations as part of their work in topology, especially algebraic topology. Their goal was to understand natural transformation for which the notion of functors and that of categories were exploited. Category theory can also be seen in some sense as a continuation of the work of Emmy Noether, one of MacLane's teachers, on formalizing *abstract processes*. Noether realized that in order to understand a type of mathematical structure, one needs to understand the processes preserving that structure. In order to achieve this understanding, Eilenberg and MacLane proposed an axiomatic formalization of the relation between structures and the processes preserving them. This indicates that category theory originally arose in mathematics out of the need of formalism to describe the passage from one type of mathematical structure to another. Initially, it was not clear whether the theory of category would turn out to be more than just a convenient language which indeed was the case for about ten years. This perspective altogether changed when categories started getting used in homology theory and homological algebra. In this connection, Maclane and Grothendieck independently described categories in which the collections of morphisms between two fixed objects were shown to have an additional structure. More specifically, it was shown that given two objects X and Y of a category C, the set Hom(X,Y) of morphisms form an abelian group [1].

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Also for reasons related to the ways homology and cohomology theories were linked, the definition of a category had to satisfy an additional formal property which led to the definition of a category that is now being commonly used. In the 1960s, Lambek proposed to describe categories as *deductive* systems. He begins with the notion of a graph consisting of two classes, *Arrows* and O*bjects*, and two mappings between them, s: Arrows $\rightarrow$ Objects and t: Arrows $\rightarrow$ Objects, namely the source and the target mappings. The arrows are usually called the *oriented edge* and the objects *nodes* or *vertices*.

With these developments, category theory became an autonomous field of research. Indeed, along with its rapid growth as a branch of mathematics, after the appearance of Lawvere's Ph.D. thesis in universal algebra, it started getting used in computer Science. However, it still remains to be seen whether category theory should be taken as an alternative to set theory. Briefly, a category is an algebraic structure consisting of a collection of *objects*, linked together by a collection of *arrows* (*morphisms*) that have two basic properties: the ability to compose the arrows associatively and the existence of an identity arrow for each object. *Objects* and *arrows* may be comprehended as abstract entities.

In fact many branches of modern mathematics can be described in terms of categories; for example, category of sets, category of relations, category of groups, etc., and most importantly, doing so often reveals deep insights and similarities between seemingly different areas of mathematics. The study of categories can be seen as an attempt to axiomatically comprehend what is characteristically common in various classes of related mathematical structures by way of relating those exploiting structure-preserving functions between them.

In view of the fact that a computer is not good at viewing concrete diagrams, category theory is being extensively used in computer science mainly because it offers a constructive mathematical structure to describe an object. Category theory has come to occupy a central position in most of the branches of mathematics, some areas of theoretical computer science where they correspond to *types* (a data type is a set of data with values having predefined characteristics like integer or floating point, usually a limited number of such data are build into a language and this corresponds to a category), and mathematical physics where categories are used to describe *vector spaces.* The notion of category generalizes those of a preorder and monoid and as well provides unification within set theoretical environment, thereby organizing and unifying much of mathematics. By now it has emerged as a powerful language or a conceptual framework providing tools to characterize the universal components of a family of structures of a given kind and their relationships. Summarily, category theory can be regarded as a mathematical theory of structures.

In addition, we invariably encounter with systems which contain objects with repeated elements or attributes (for example, groups of people, systems of elementary particles, etc., having two or more elements with the same property). We need a (formal) mathematical structure to model this kind of data. In the recent years such mathematical structures have been developed which are in general called *multiset-based structures*. Note that a *multiset* is a collection of objects in which repetition of elements is considered significant. Accordingly, sets are merely special instances of multisets.

Applications of multisets abound, especially in mathematics and computer science [2], [3]. Typically, multisets have been used in data base theory, membrane computing, etc. For example, in membrane computing, each membrane can be viewed as a collection of objects appearing in multiple copies.

Recently, considering the applications of multisets, on the one hand, and that of categories on the other, categories of multisets are being studied.

II. The Development of The Theory of Categories

Categories were first introduced in course of formulating algebraic topology, specifically with Samuel Eilenberg's observation that Saunders Maclane's calculations on a specific case of a group extension coincided precisely with Norman Steenrod's calculation of the homology of a *solenoid*. Eilenberg and Maclane's effort to make sense of this coincidence across apparently distinct areas of mathematical inquiry gave rise to introducing category theory. The central notion at work was that of *natural transformations*. In order to provide a broad mathematical perspective, the notion of *functor* was introduced for which they borrowed the term *category* from the philosophical writings of Aristotle, Kant and reiterated in C. S. Peirce.

Emmy Noether (one of Maclane's teachers), in formalizing abstract processes, realized that understanding of a mathematical structure in its proper perspective could be better achieved through a proper understanding of the processes preserving that structure. Maclane and Eilenberg proposed an axiomatic formalization of the relation between structures and the processes preserving them, which is considered as a first sustained formalization of Noether's intuitive notion of the concept of category.

In 1945, Eilenberg and Maclane define *category C* as an aggregate *Ob* of abstract elements, called the *objects* of C and abstract elements *Map*, called *mappings* of the category. The term *Map* is characterized as follows:

(C1) Given three mappings α_1, α_2, α_3, the *triple product* $\alpha_3(\alpha_2\alpha_1)$ is defined if and only if $(\alpha_3\alpha_2)\alpha_1$ is defined. That is, whenever either is defined, the associative law $\alpha_3(\alpha_2\,\alpha_1) = (\alpha_3\,\alpha_2)\,\alpha_1$ holds. This triple product is often written as $\alpha_3\alpha_2\alpha_1$.

(C2) The triple product $\alpha_3\alpha_2\alpha_1$ is defined whenever both the products $\alpha_3\alpha_2$ and $\alpha_2\alpha_1$ are defined.

(C3) For each mapping α, there is at least one identity ℓ_1 such that $\alpha\ell_1$ is defined and at least one identity ℓ_2 such that $\ell_2\alpha$ is defined.

(C4) The mapping ℓ_x corresponding to each object X is an identity.

(C5)For each identity ℓ there is a unique object X of **C** such that $\ell_X = \ell$.

It is remarked that objects play a secondary role and could be entirely done away from the definition; however, it would make the manipulation of the applications less convenient. In fact the definition of category formulated by Eilenberg and Maclane emerged as a helping tool to provide an explicit and rigorous formulation of the notions of functors and natural transformations.

The theory of category which is called a *mathematical universe* was developed from some basic definitions of maps, composition and algebra of composition, and further defined as a system consisting of:

(i) Objects and maps.

(ii) For each map f, one object as domain and one object as codomain.

(iii) For each object A, an identity map consisting A as domain and codomain.

(iv) A composite map gf: $A \to C$ for each pair of maps f: $A \to B$, g: $B \to C$ satisfies identity and associativity laws [4].

Category theory interacts with nearly everything; it is a remarkable empirical fact that the important structural properties of mathematical objects are often expressible in category-theoretic terms, specifically as a universal property. For better understanding of category theory, a survey on the ways category theory interacts with set theory was carried out. It is observed that many of the elementary concepts of category theory were introduced for the purpose of expressing familiar concepts of set theory and their generalizations in other areas of mathematics [5]. Categories were mostly described in terms of objects and arrows. On another view which is interesting, it is observed that morphism is the central concept in a category, and a category C is defined as consisting of the collection Mor_C of the morphisms of C. Objects of C are associated with identity morphisms 1_A, since 1_A is unique in each set $\mathrm{Mor}_C(A, A)$ and uniquely identifies the object A [6]. On a similar view the theory is described as a collection of maps which have a partial associative multiplication and a system of units. This indicates that the objects are actually redundant structure and their role can be replaced by the identity maps. It is further remarked that categories and functors form a category called Cat [7]**.** In another development, a category C is defined as a graph together with two functions c: $C_2 \to C_1$ and u: $C_0 \to C_1$. The elements of C_0 are called *objects* and those of C_1 are called *arrows*. The function c is called *composition* and if (g,f) is a composable pair, c(g,f) is written $g \circ f$ and is called the *composite* of g and f. If A is an object of **C**, u(A) is denoted id_A, called the *identity* of the object A.

The source of $g \circ f$ is the source of f, and the target of $g \circ f$ is the target of g.

The following hold:

$(h \circ g) \circ f = h \circ (g \circ f)$, whenever either side is defined.

The source and target of id_A are both A.

If f: $A \rightarrow B$, then $f \circ id_A = id_B \circ f = f$.

Functional Programming Languages are described as Categories [8]. Category theory is being studied from applications point of view, specifically within *algorithmics (problem solving)*. It is observed that the language of category theory facilitates providing an elegant style of describing *expressions* and *proof (equation reasoning)* for use in algorithmic. This happens to be reasoning at the functional level without the need (and the possibility) to reduce arguments explicitly. The equational formulas often lead to a far-reaching generalization much more than the usual set-theoretic formulations. A category is defined as a *data* characterized as follows:

III. The Data

(i) A collection of things called *objects*. By default, A, B, C,...,X,Y,Z,... vary over objects.

(ii) A collection of things called *morphisms*, sometimes called *arrows*. By default, f, g, h, ..., and later on, also α, β, φ, ψ, χ, ..., vary over morphisms. The collection of all arrows of a category C is sometimes denoted ArrC.

(iii) A relation on morphisms and pairs of objects called *typing* of the morphisms. By default, the relation is denoted f: $A \rightarrow B$ for morphism f and objects A, B. Here, A is the domain and B the codomain of f: $A \rightarrow B$.

(iv) A binary partial operation on morphisms called *composition*. By default, f; g is the notation of the composition of morphisms f and g. An alternative notation is $g \circ f$ or gf.

(v) For each object A, a distinct morphism called *identity* on A. By default, id_A, or id when A is clear from the context denotes the identity on object A.

IV. The Axioms

There are three typing axioms and two axioms for *equality*. The typing axioms:

(T_1) f: $A \rightarrow B$ and f: $A' \rightarrow B'$ implies

$A = A'$ and $B = B'$. (Unique – Type)

(T_2) $f : A \rightarrow B$ and $g : B \rightarrow C$ implies

$f ; g : A \rightarrow C$. (Composition – Type)

(T_3) id_A: $A \rightarrow A$. (Identity – Type)

A morphism term f is called *well-typed* if, a typing

f: $A \rightarrow B$ can be derived for some objects A, B according to the aforesaid axioms.

V. Axioms for Equality of Morphisms

(E_1) $(f ; g) ; h = f ;(g; h)$

(Composition –Assoc.)

(E_2) $id ; f = f = f ; id$ (Identity)

Also, whenever a term is written it is assumed that the variables are typed (at their introduction-mostly an implicit *universal quantification* in front of the formula) in such a way that the term is well-typed. This allows us to simplify the formulations considerably, as illustrated in the axioms for equality of morphisms above [9].

A category K is also defined as a pair (ob(K), mor(K)) of generic objects A,B,... in ob(K) and generic arrows f: $A \rightarrow B$, g: $B \rightarrow C$,... in mor(K) between objects, with associative composition and identity (loop) arrow. It is observed that the theory was born with an observation that many properties of mathematical systems could be unified and simplified by a presentation with commutative diagrams of arrows [10]. A Category **C** is defined as consisting of the following:

(i)a collection Obj_C of objects A,B,C, ... , X,Y,Z, ...,

(ii)a collection **C**(A,B) of arrows f: $A \rightarrow B$ from A to B for each pair of objects A and B.,

(iii)for each object A, an identity arrow $id_A : A \rightarrow A$.,

(iv)for each pair of arrows $f : A \rightarrow B$, $g: B \rightarrow C$, a composite arrow $g \circ f$: $A \rightarrow C$. These data satisfy the following generalised *monoid laws:*

1. Identity: If f: $A \rightarrow B$ then $id_B \circ f = f \circ id_A$

2. Associativity: If f: $A \rightarrow B$, g: $B \rightarrow C$ and h: $C \rightarrow D$ then

$(h \circ g) \circ f = h \circ (g \circ f)$.

The theory is considered as the mathematical study of universal properties and that, it brings to light, makes explicit, and abstracts out the relevant structures often hidden in following traditional approaches. It also looks for the universal properties holding in the categories of structures one is working with. A monoid is a category with only one object and a pre-order is a category with at most one arrow between every two objects are most simple examples of categories [11]. Every directed graph can be made into a category where the objects are the vertices of the graph and the arrows are paths in the graph [12]. A Poset (P, $\leq$) can be regarded as a category $\mathfrak{P}$ with the set of object P, along with the provision that if $x, y \in P$, then $\mathfrak{P}(x, y)$ consists of exactly one morphism if $x \leq y$ and is empty otherwise. Moreover, this shows that every set can be viewed as a discrete category i.e., a category where the only morphisms are the identity morphisms [13]. The theory is also defined as consisting of the following:

(i) a class Ob(C) of objects of C.,

(ii) a family Mor(C) associating with every pair $A,B \in Ob(C)$, a class Mor(C)(A,B) of morphisms from A to B.,

(iii) for all $A,B,C \in Ob(C)$, a mapping $o_{A,B,C}$: $Mor(C)(B,C) \times Mor(C)(A,B) \rightarrow Mor(C)(A,C)$, called *composition.,*

(iv) for $A \in Ob(C)$ a distinguished morphism $id_A \in Mor(C)(A,A)$, called the *identity* morphism for A. These data satisfy the following:

1.For all $A,B,C \in Ob(C)$ and $f \in Mor(C)(A,B)$, $g \in Mor(C)(B,C)$ and $h \in Mor(C)(C,D)$,

$h \circ (g \circ f) = (h \circ g) \circ f$

i.e., $o_{A,C,D}(h.o_{A,B,C}(g.f)) = o_{A,B,D}(o_{B,C,D}(h.g).f)$.

2.For all $A,B,C \in Ob(C)$ and $f \in Mor(C)(A,B)$ and $g \in Mor(C)(C,A)$, $f \circ id_A = f$ and $id_A \circ g = g$ i.e., $o_{A,A,B}(f.id_A) = f$ and $o_{C,A,A}(id_A.g) = g$.

Further, it is elaborated that a category whose objects are sets, whose morphisms from A to B are the set-theoretic functions from A to B and where composition is given by $(g \circ f)(x) = g(f(x))$, is denoted Set. It follows that in Set, the identity morphism id_A takes every $x \in A$ to itself [14].

Using the usual first-order theory of category, eight first-order axioms were adjoined to obtain an elementary theory of the category of sets which was shown to provide a foundation for mathematics, which is quite different from the usual set-theoretic formulation in the sense that much of number theory, elementary analysis, and algebra can be constructively developed within it [15]. In the process of describing the *Grothendieck Universe* (or just a *universe*), a category is defined as an algebraic object similar to a group or a ring. It is further elaborated that if we desire to talk about the category of *all sets*, *all groups*, *all topological spaces*, along with not allowing the paradox of *too large sets* (Russell's Paradox) to appear, we need to introduce *fancy* set-theoretic ideas (like Grothendieck Universes: the full second order version of cumulative hierarchy) [16]. The connection between programming and category theory was illustrated while describing computational category theory. A category is defined as a graph (O,A,s,t) whose nodes *O* are called *objects* and edges *A arrows*. Associated with each object a in O, there is an arrow

i_a:a $\rightarrow$ a, the identity arrow a, and to each pair of arrows f: a $\rightarrow$ b and g:b $\rightarrow$ c, there is an associated arrow gf: a $\rightarrow$ c, the composition of f with g. The following equations must hold for all objects a, b, c and d, and arrows f:a $\rightarrow$ b, g:b $\rightarrow$ c and h: c $\rightarrow$ d, *(hg)f* = h(gf) and $fi_a = f = i_b f$. It is observed that category theory seems to operate on the same level of generality as logic and computer programming. The essential virtue of category theory is to provide suitable means for making definitions which is the programmer's main task. Computer scientists are fascinated by category theoretic approaches because it is largely *constructive*. Theorems asserting the existence of objects are proven by explicit constructions. In this sense, category theory may be viewed as a *warehouse* of algorithms [17]. Category theory provides a means for organizing and classifying the relationships between mathematical concepts in mathematical models. It is emphasized that the theory of category is the language of mathematical concepts and relations and thereby making it the language of mathematics [18]. There are few contributions on category theory in multiset context. The first was in 1987 where multiset and multiset functions were noticed to determine a category naturally [19]. After sixteen years, a categorical model of multisets was described and, more so the idea that multiset and multiset morphisms form a category, denoted Mul was further elaborated [20].

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